A Novel Holomorphic Form of the Collatz Conjecture

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Abstract

This paper describes some functional equations which occur connected with the famous Collatz Conjecture. In particular, it discusses two functional equations of the Collatz Problem. Using the Roots of Unity similar modular functions are generalized into functional equations.

1 Preliminaries

The Collatz Conjecture is formally defined as follows. Given $n \in \mathbb{Z}^+$, define a function f such that

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

We proceed to define $f^k(n)$ as the k^{th} composition of $f(n)$ i.e

$$\underbrace{f \circ \cdots \circ f}_{\text{k compositions}} = f^k(n)$$

The Collatz Conjecture can then be written as:

For all positive integers n there exists finite k such that $f^k(n) = 1$

2 Functional Equations

We begin by attempting to convert the piece-wise function initially stated to a more standard function defined without the cases.

Define a function g such that

$$g(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2 \\ 1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

One such function is shown:

$$g(n) = \frac{1}{2}\left((-1)^{n+1} + 1\right)$$

We can now define our initial function $f(n)$ in terms of $g(n)$ like so:

$$\begin{aligned} f(n) &= \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases} \\ &= g(n) \cdot \left(\frac{3}{n} + 1\right) + g(n-1) \cdot \left(\frac{n}{2}\right) \\ &= \left(\frac{1}{2}\left((-1)^{n+1} + 1\right)\right)(3n+1) + \left(\frac{1}{2}\left((-1)^{n} + 1\right)\right)\left(\frac{n}{2}\right) \\ f(n) &= \frac{1}{4}\left(7n + 2 + (-1)^{n+1}(5n+2)\right) \qquad \text{i} \end{aligned}$$

2.1 Holomorphic Equations

If we assume n is real, we can write the equation similarly but using complex numbers:

$$\frac{1}{4}\left(2 + 7n - (2+5n)\cos(n\pi)) - \frac{1}{4}i(2+5n)\sin(n\pi)\right)$$

This can then be simplified using Euler's formula:

$$\frac{1}{4}\left(2 + 7n - e^{in\pi}(2+5n)\right)$$